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Research Paper

INVARIANTS OF THE PLANE ELASTICITY TENSORS

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A new technique is used to find the set of five well known polynomial invariants, under O(2), of the space of plane elasticity tensors. An elementary argument shows that this set forms an integrity basis for the space.

Keywords: Linear elasticity, Plane tensors, Integrity basis

INTRODUCTION

Let $\mathbb{E}la$ denote the vector space of *two-dimensional* elasticity tensors C. With reference to a basis, a member of this space is denoted by C_{ijkl}, where the subscripts take values 1 or 2 only and the following usual symmetry conditions are satisfied,

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}.$$

Let $O(2)$ be the group of orthogonal transformations on $\mathbb{R}^2$. A function ¨(C) defined on Ela is said to be invariant if ¨(C) = ¨$(Q * \mathrm{C})$ where for $Q \in O(2)$ and $\mathrm{C} \in$ Ela

$$(Q * \mathrm{C})_{pqrs} = Q_{pi} Q_{qj} Q_{rk} Q_{sl} C_{ijkl}.$$

A finite set of invariants is called a *functional basis* if every invariant can be expressed in terms of members of this set. A finite set B of polynomial invariants is called an *integrity basis* if every polynomial invariant can be expressed as a polynomial function in terms of members of B.

Let

$$c_{11} = C_{1111},\ c_{12} = C_{1122},\ c_{13} = \sqrt{2}\, C_{1112},$$
$$c_{22} = C_{2222},\ c_{23} = \sqrt{2}\, C_{2212},\ c_{33} = 2\, C_{1212}.$$

Several authors, using different techniques, have found the following five polynomial invariants, or equivalent, of the space of plane elasticity tensors (Zheng, 1994; Blinowski *et al.,* 1996; Vianello, 1997; Vannuchi, 2005; and de Saxe *et al.,* 2013).

$$I_1 = c_{11} + c_{22} + 6c_{12} - 2c_{33},$$
$$I_2 = c_{11} + c_{22} - 2c_{12} + 2c_{33},$$

$$I_3 = \frac{1}{2}(c_{11} - c_{22})^2 + (c_{13} + c_{23})^2,$$

$$I_4 = \frac{1}{8}[(c_{11} + c_{22}) - 2c_{12} 2c_{33}]^2 + (c_{13} - c_{23})^2,$$

$$I_5 = (c_{11} + c_{22} - 2c_{12} - 2c_{33})[(c_{11} - c_{22})^2 - 2(c_{13} + c_{23})^2] + 8(c_{13}^2 - c_{23}^2)(c_{11} - c_{22})..$$

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Vianello (1997) has proven that the set $\{I_1, I_2, .., I_5\}$ is an integrity basis for the space Ela.

In this brief note, we use a technique, due to Dieulesaint and Royer (2000), which relies on the diagonalization of the rotation matrix. It furnishes us with a valuable criterion in that a monomial in the transformed space will be an invariant if and only if the number of ones in the subscripts equals the number of twos. This not only facilitates evaluation of the above five invariants but also provides an elementary proof of the integrity theorem. It is also hoped that the technique presented in the present paper will be helpful in treating the, as yet, unsolved problem of finding an integrity basis for the space of three-dimensional elasticity tensors, see de Saxe *et al.* (2013).

INVARIANTS OF PLANE TENSORS

A tensor function defined on $\mathbb{E}la$ will be invariant under $O(2)$ provided it is invariant under the action of $SO(2)$ followed by a reflection with respect to the basis vector, $\mathbf{e}_1$. Transformation matrix $Q(\theta)$ for a rotation through an angle θ is,

$$Q(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Matrix $\tilde{Q}$ representing the reflection about $\mathbf{e}_1$ is,

$$\tilde{Q} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

To proceed further, we need to diagonalize $Q(\theta)$. Same technique was employed by Ahmad and Rashid (2009) to investigate invariants of the elasticity tensors under $SO(3)$. Matrix $Q(\theta)$ has eigenvalues $e^{i\theta}, e^{-i\theta}$ with respective eigenvectors,

$$\tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \quad \tfrac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}.$$

These eigenvectors become new basis vectors with respect to which now complex elastic moduli γ_{ijkl} are given by the transformation equation,

$$\gamma_{ijkl} = b_{pi} b_{qj} b_{rk} b_{sl} C_{pqrs},$$

where b_{pi}, etc., are elements of the matrix B,

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

Rotation matrix $D(\theta)$ has become diagonal,

$$D(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

Elastic moduli γ_{ijkl} before and after rotation are related by,

$$\gamma'_{ijkl} = e^{(\nu_1 - \nu_2)\theta} \gamma_{ijkl},$$

where

ν_1 = number of ones among *ijkl,*

ν_2 = number of twos among *ijkl.*

Hence γ *w*ill be invariant if and only if, among the subscripts, the number of ones and twos are equal. We immediately get the following set of invariants.

$$T_1 = \gamma_{1122},\ T_2 = \gamma_{1212},\ T_3 = \gamma_{1111}\gamma_{2222},$$

$$T_4 = \gamma_{1222}\gamma_{2111},\ T_5 = \gamma_{2111}^2 \gamma_{2222} + \gamma_{1222}^2 \gamma_{1111}.$$

Above invariants are essentially the same as $I_1, I_2, .., I_5$ as indicated by the following relations,

$$T_1 = -\frac{1}{4} I_2,$$

$$T_2 = -\frac{1}{8}(I_1 + I_2),$$

$$T_3 = \frac{1}{2} I_4,$$

$$T_4 = \frac{1}{8} I_3,$$

$$T_5 = -\frac{1}{32} I_5.$$

A monomial M in components of ¥ will consist of factors of the form $¥^{p}_{ijkl}$ and will transform under $SO(2)$ to $\exp\{i(\sum_k \neg_1^{(k)} - \sum_k \neg_2^{(k)})^{-}\}M$ where $\neg_1^{(k)}$ and $\neg_2^{(k)}$ respectively denote the number of ones and twos in the $k-th$ factor. Thus a monomial will be invariant under $SO(2)$ if and only if the total number of ones and twos is equal. In the laboratory frame sum of two or more components of C may be invariant while separately they may not be such. However, in the transformed frame, each term of a polynomial invariant must be invariant on its own.

From the transformation matrix (3), it is clear that the following relations exist among b_{pq},

$$b_{2q} = ib^*_{1q},\ q = 1, 2,$$

$$b_{1q} = ib^*_{2q},\ q = 1, 2.$$

where star denotes complex conjugation. The above relations lead to the conclusion that an interchange, $1 \leftrightarrow 2$, among the subscripts of any component of ¥ yields its conjugate. For example,

$$
\begin{aligned}
¥_{1222} &= b_{1p} b_{2q} b_{2r} b_{2s} C_{pqrs},\\
&= i^4 b^*_{2p} b^*_{1q} b^*_{1r} b^*_{1s} C_{pqrs},\\
&= ¥^*_{2111}.
\end{aligned}
$$

Thus all invariants $T_1, .., T_5$ are real. The above argument has established invariance with respect to $SO(2)$. However since $\tilde{Q}\mathbf{e}_2 = -\mathbf{e}_2$, an even number of twos in the subscripts of invariants ensures invariance with respect to $O(2)$.

Some of the above results are similar to the ones obtained by Verchery and Vannucci (Verchery, 1979; Vannuchi, 2005; and Vannuchi and Verchery, 2010). This method starts with the transformation of a vector $\mathbf{x} = (x, y)$ to a complex contravariant vector (X^1, X^2) as

$$X^1 = \frac{1}{\sqrt{2}} e^{-\frac{i®}{4}} z = \frac{x + y - i(x - y)}{2},$$

$$X^2 = \frac{1}{\sqrt{2}} e^{\frac{i®}{4}} \bar{z} = \frac{x + y + i(x - y)}{2}.$$

The above transformation is interpreted as a change of frame with the transformation matrix given by

$$\begin{pmatrix} \frac{1-i}{2} & \frac{1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{pmatrix}$$

The contravariant components of the elasticity tensor transform under a rotation governed by a diagonal matrix which easily leads to invariants equivalent to the ones found above. Thus the two techniques are similar in that they introduce a complex change of frame to make the rotation matrix diagonal. This is rewarded by the fact that a single component of the elasticity tensor in the new frame represents a polynomial in the laboratory frame. This fact plays a crucial role in the next section when we combine it with our result that any monomial in the transformed frame is an invariant provided the subscripts are evenly divided among ones and twos.

INTEGRITY BASIS

We shall show that the set, T, of five polynomial invariants $\{T_1, T_2, T_3, T_4, T_5\}$ forms an integrity basis for Ela under the group $O(2)$. It is clear that all invariants of degree up to and including 3 are expressible in terms of members of I. We shall prove the general result by induction. Suppose the following statement is true.

Every polynomial invariant of Ela under $O(2)$ of degree up to and including k is expressible as a polynomial in terms of members of T.

Now an invariant of degree $k+1$ must contain an expression with $k+1$ factors $¥_{ijkl}$ with various permutations of $ijkl$ with the stipulation that half of the total number of indices will be ones and the other half will be twos. If an interchange of, $1 \leftrightarrow 2$, does not leave it unchanged, we add to it an expression found by this interchange. The result will be a real invariant of degree $k+1$. For example if $k=5$ then we can form an invariant of degree 6 in the following manner.

$$¥_{1112}^{2}¥_{2211}¥_{2111}¥_{1222}¥_{2222}$$
$$+¥_{2221}^{2}¥_{1122}¥_{1222}¥_{2111}¥_{1111}.$$

Since $¥_{1122} = ¥_{2211}$, we can factorize the above expression as,

$$¥_{1122}(¥_{1112}^{2}¥_{2111}¥_{1222}¥_{2222}$$
$$+¥_{2221}^{2}¥_{1222}¥_{2111}¥_{1111}),$$

$$= T_1(\text{an invariant of degree} 5.)$$

By our assumption, the expression in parenthesis can be expressed in terms of members of T, therefore the result holds so far as the above example is concerned. We shall apply a similar technique for the general case. There are various possibilities.

- One of the factors in the first part contains equal number of ones and twos, i.e., it is T_1 or T_2. In this case, the invariant of degree $k+1$ will become a product of T_1 (or T_2) and an invariant of degree k. Hence the result holds in this case.
- The invariant T_{k+1} is of the form,

  $$T_{k+1} = ¥_{1112}^{p}¥_{2111}^{q}¥_{1111}^{r}¥_{2222}^{s} + c.c,$$

  where p, q, r, s are nonnegative integers. If both p and q are different from zero then it is possible to write $T_{k+1} = T_4(\text{an invariant of degree } k-1)$. If both r and s are different from zero, then $T_{k+1} = T_3(\text{an invariant of degree } k-1)$.
- The only remaining case is

  $$T_{k+1} = ¥_{1112}^{p}¥_{2222}^{s} + ¥_{2221}^{p}¥_{1111}^{s}.$$

  Since the degree of the invariant is $k+1$, we have,

  $$p+s = k+1.$$

  Also the number of ones must be one half the total in the first part. This leads to,

  $$3p = 2(k+1).$$

  Hence we have,

  $p = 2s = \dfrac{2(k+1)}{3}$. An invariant of this form will exist only if 3 is a divisor of the degree $k+1$. Let $s=m, p=2m$ where $m>1$ is a positive integer. Now

$$T_{3m} - T_5^{m} = \sum_{i=1}^{m-1}[¥_{1112}^{2}¥_{2222}]^{m-i}[¥_{2221}^{2}¥_{1111}]^{i}.$$

From the right side of Equation (8) a factor $¥_{1112}^{2}¥_{2221}^{2}¥_{1111}¥_{2222} = T_3T_4^2$ can be taken out leaving an invariant of degree $3m-6 = k-5$ which by assumption is a polynomial function of members of T.

Thus the set $\{T_1, T_2, .., T_5\}$ forms an integrity basis for $\mathbb{E}la$.

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