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Research Paper

# SOLUTIONS TO SINGULAR BOUNDARY – VALUE PROBLEMS USING RECURSIVE FORM OF QUADRATIC B-SPLINE COLLOCATION METHOD

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A B-spline collocation method is by using the recursive form of B-spline basis functions as basis. The method is to solve second order singular differential equations with Dirchlet's boundary conditions. Results of Numerical examples show the efficiency of the method. Stability of present method and accuracy of numerical solution is improved by decreasing the nodal space.

**Keywords:** Collocation method, B-splines, Singular differential equations

## INTRODUCTION

In recent years, many numerical methods are evolved to solve singular differential equations with Neumann-Dirchlet's boundary conditions. The methods include like Finite difference method (Pandey and Arvind, 2004), kernel space (Geng and Cui, 2007; Li *et al.*, 2012) sinc collocation method (Mohsen, 2008), B-spline collocation method (Hikmet, 2006) and predictor and corrector method (Abdalkaleg, 2014) and many more. The B-spline based collocation method is applied to evaluate boundary value problems including singular boundary value problems (Joan, 2009).

However, it is observed that B-spline basis functions are derived using fixed equidistant space for a particular degree only. If the recursive formulation given by Carl. De boor (1982) is used, the basis function evaluation can be generalized and any degree of the basis function can be used in collocation method for uniform or non uniform mesh sizes.

In this paper, after defining the B-spline basis function recursively, the B-spline collocation method is described and formulated. The efficiency of the method is demonstrated using the second order singular differential equations with Dirchlet's boundary conditions.

Considering second order linear differential equations with constant coefficients

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$$\frac{d^2U}{dx^2} + k_1 P(x) \frac{dU}{dx} + k_2 Q(x)U = R(x),$$

$$a \leq x \leq b \quad \dots(1)$$

with the boundary conditions

$$U(a) = d_1, U(b) = d_2$$

where a, b, d<sub>1</sub>, d<sub>2</sub>, k<sub>1</sub> and k<sub>2</sub> are constant P(x), Q(x) and R(x) are functions of

$$\text{Let } U^k(x) = \sum_{i=2}^{n-1} C_i N_{i,p}(x) \quad \dots(2)$$

where C<sub>i</sub>'s are constants to be determined and N<sub>i,p</sub>(x) are B-spline basis functions, be the approximate global solution to the exact solution U(x) of the considered second order singular differential equation (1).

### 1B-SPLINES

In this section, definition and properties of B-spline basis functions (Hughe, 2002; David) are given in detail. A zero degree and other than zero degree B-spline basis functions are defined at x<sub>i</sub> recursively over the knot vector space

$$X = \{x_1, x_2, x_3, \dots, x_{n-1}, x_n\} \text{ as}$$

i) if p = 0

$$N_{i,p}(x) = 1 \quad \text{if } x \in (x_i, x_{i+1})$$

$$N_{i,p}(x) = 0 \quad \text{if } x \notin (x_i, x_{i+1})$$

ii) if p ≥ 1

$$N_{i,p}(x) = \frac{x - x_i}{x_{i+p} - x_i} N_{i,p-1}(x) + \frac{x_{i+p+1} - x}{x_{i+p+1} - x_{i+1}} N_{i+1,p-1}(x)$$

where p is the degree of the B-spline basis function and x is the parameter belongs to X. When evaluating these functions, ratios of the form 0/0 are defined as zero

### Derivatives of B-splines

If p=2, we have

$$N'_{i,p}(x) = \frac{x - x_i}{x_{i+p} - x_i} N'_{i,p-1}(x) + \frac{N_{i,p-1}(x)}{x_{i+p} - x_i}$$

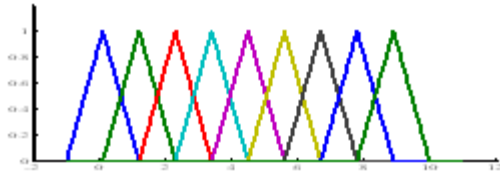
$$+ \frac{x_{i+p+1} - x}{x_{i+p+1} - x_{i+1}} N'_{i+1,p-1}(x) - \frac{N_{i+1,p-1}(x)}{x_{i+p+1} - x_{i+1}}$$

$$N'_{i,p}(x) = 2 \frac{N'_{i,p-1}(x)}{x_{i+p} - x_i} - 2 \frac{N'_{i+1,p-1}(x)}{x_{i+p+1} - x_{i+1}} \quad \dots(4)$$

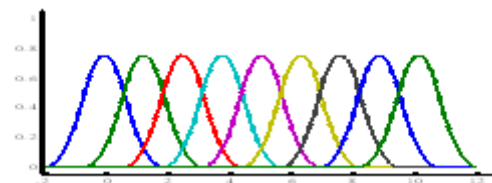
In the above equations, the basis functions are defined as recursively in terms of previous degree basis function, i.e., the p<sup>th</sup> degree basis function is the combination of ratios of knots and (p-1) degree basis function. Again (p-1)<sup>th</sup> degree basis function is defined as the combination ratios of knots and (p-2) degree basis function. In a similar way every B-spline basis function of degree up to (p-(p-2)) is expressed as the combination of the ratios of knots and its previous B-spline basis functions.

The B-spline basis functions are defined on knot vectors. Knots are real quantities. Knot vector is a non decreasing set of Real numbers. Knot vectors are classified as non-uniform knot vectors, uniform knot vector and open uniform knot vectors. Uniform knot vector in which difference of any two consecutive knots is constant is used for test problems in this paper. Two knots are required to define the zero degree basis function. In a similar way, a p<sup>th</sup> degree B-

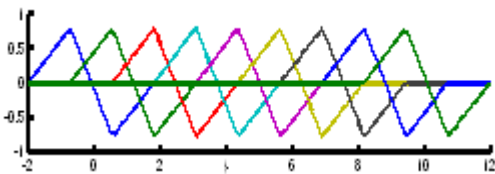
**Figure 1: First Degree B-spline Basis Function with Uniform Knot Vector  $X = \{-1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$**



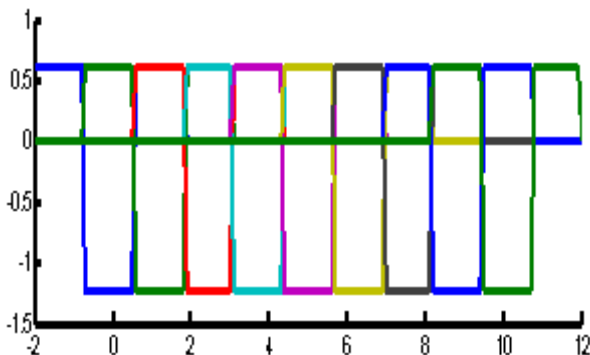
**Figure 2: Second Degree B-spline Basis Function with Uniform Knot Vector  $X = \{-2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$**



**Figure 3: First Derivative of Second Degree B-spline Basis Function with Uniform Knot Vector  $X = \{-2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$**



**Figure 4: Second Derivative of Second Degree B-spline Basis Function with Uniform Knot Vector  $X = \{-2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$**



spline basis function at a knot have a domain of influence of  $(p+2)$  knots. B-spline basis functions of degree one and degree two over uniform knot vector are shown graphically below in Figures 1 and 2. The first and second order derivatives of B-spline base function is presented graphically in Figure 3 and in Figure 4 respectively.

**B-spline Collocation Method**

Collocation method is widely used in approximation theory particularly to solve differential equations. In collocation method, the assumed approximate solution is made it exact at some nodal points by equating residue zero at that particular node. B-spline basis functions are used as the basis in B-spline collocation method whereas the base functions which are used in normal collocation method are the polynomials vanishes at the boundary values. Residue which is obtained by substituting equation (2) in equation (1) is made equal to zero at nodes in the given domain to determine unknowns in (2). Let  $[a, b]$  be the domain of the governing differential equation and is partitioned as  $n$  sub domains with equal

length  $h = \frac{b-a}{n}$  of  $n$  sub domains. The

are known as nodes, the nodes are treated as knots in collocation B-spline method where B-spline basis functions are defined and these nodes are used to make the residue equal to zero to determine unknowns  $C_i$ 's in (2). Two extra knot vectors are taken into consideration beside the domain of problem both side when evaluating the second degree B-spline basis functions at the nodes.

Substituting, the approximate solution (2) and its derivatives in (1).



$$\frac{d^2U^h}{dx^2} + k_1P(x)\frac{dU^h}{dx} + k_2Q(x)U^h = R(x) \text{ i.e.}$$

$$\sum_{i=-2}^{n-1} C_i N''_{i,p}(x) + k_1P(x) \sum_{i=-2}^{n-1} C_i N'_{i,p}(x) + k_2Q(x) \sum_{i=-2}^{n-1} C_i N_{i,p}(x) = R(x) \quad \dots(5)$$

Equation (5) which is valuated at  $x_i$ 's,  $i = 0, 1, 2, \dots, n-1$  gives the system of  $(n-1) \times (n+1)$  equations in which  $(n+1)$  arbitrary constants are involved. Two more equations are needed to have  $(n+1) \times (n+1)$  square matrix which helps to determine the  $(n+1)$  arbitrary constants. The remaining two equations are obtained using

$$\sum_{i=-2}^{n-1} C_i N_{i,p}(a) = d_1 \quad \dots(6)$$

$$\sum_{i=-2}^{n-1} C_i N_{i,p}(b) = d_2 \quad \dots(7)$$

Now using all the above equations (5), (6), (7), i.e.,  $(n+1)$  a square matrix is obtained which is diagonally dominated matrix because every second degree basis function has values other than zeros only in three intervals and zeros in the remaining intervals, it is a continuing process like when one function is ending its effect in its surrounding region than other function starts its effectiveness as parameter value changing. In other words, every parameter has at most under the three ( $p=2$ ) basis functions. The systems of equations are easily solved for arbitrary constants  $C_i$ 's. Substituting these constants in (2), the approximation solution is obtained and used to estimate the values at domain points.

## NUMERICAL EXPERIMENTS

The effectiveness of the present method is demonstrated by considering the various examples

**Example 1:** The exact solution of singular boundary value problem considered below (Abdalkaleg, 2014)

$$U'' + \frac{1}{x(x-1)}U = \frac{-72}{1045}x^2 + \frac{12}{1045}x^3$$

$$+ \frac{1}{209}x^4 + \frac{1}{19}x^5, \quad U(0) = U(1) = 0$$

$$U(x) = \frac{1834592}{887331445}x + \frac{917296}{887331445}x^2$$

$$+ \frac{458648}{887331445}x^3 - \frac{188072}{34571355}x^4 + \frac{14131}{29252685}x^5$$

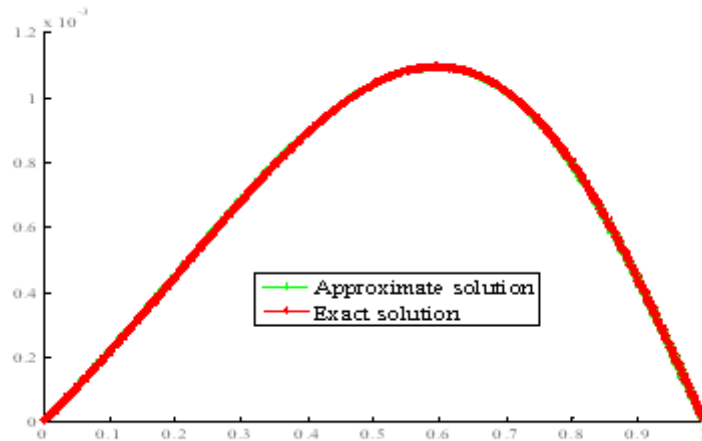
$$+ \frac{32}{278597}x^6 + \frac{1}{817}x^7$$

It is observed that the Maximum absolute Relative error is constantly decreasing as mesh size is decreasing. This shows that B-spline collocation method guarantees the convergence of the approximate solution to exact solution.

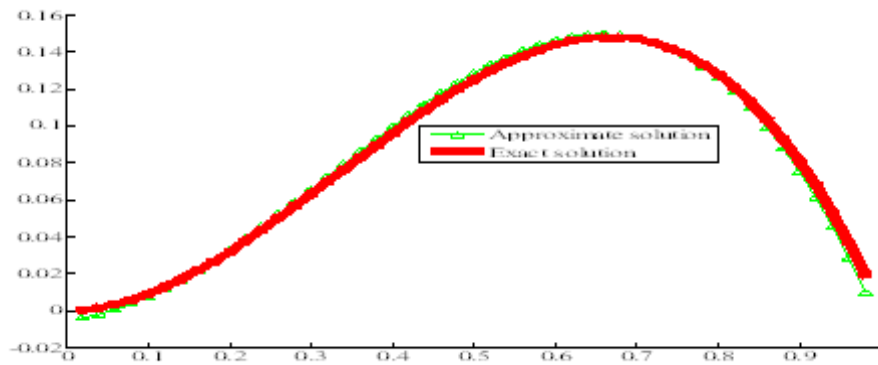
The following below Figure 5 gives the comparison of B-spline collocation solution and exact solution for example1 for mesh size 'h=0.002'. Decreasing mesh size improves the approximation solution as well and also it can be observed that it consistently moving to converge with exact values.

Mesh size(h)	0.0100	0.005	0.0025	0.0015	0.001	0.0005	0.0003
Maximum absolute Relative error	0.5080	0.5049	0.5029	0.5020	0.5014	0.5008	0.5006

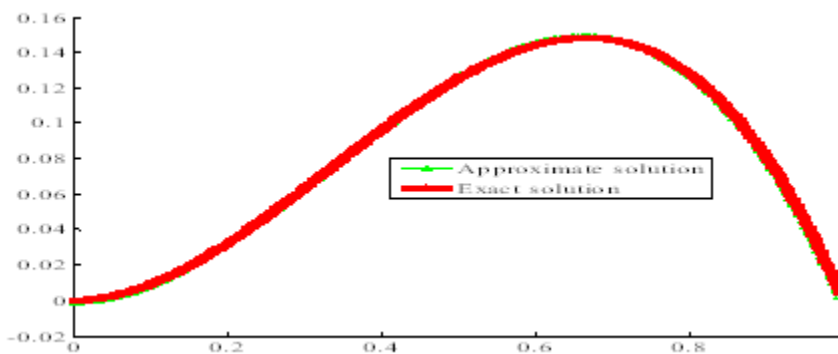
**Figure 5: Comparison of Approximate Solution and Exact Solution for the Example1 for Mesh Size h=0.002**



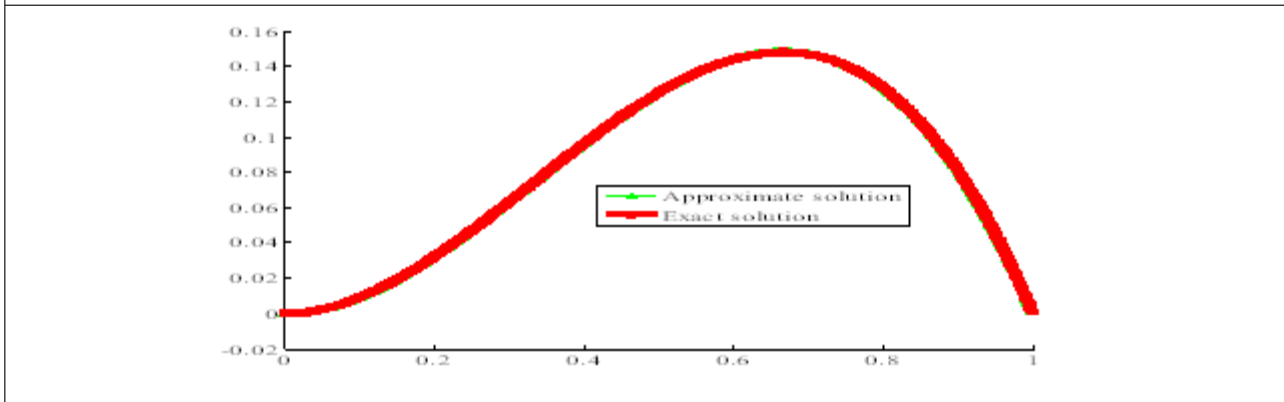
**Figure 6: Comparison of Approximate Solution and Exact Solution for the Example2 for Mesh Size h = 0.0200**



**Figure 7: Comparison of Approximate Solution and Exact Solution for the Example2 for mesh size h= 0.0040**



**Figure 8: Comparison of Approximate solution and Exact Solution for the Example2 for mesh size h= 0.0013**



**Example 2:** Consider a singular boundary value problem (Abdalkaleg, 2014) given by

$$U'' + \frac{1}{x}U' + U(x) = 4 - 9x + x^2 - x^3,$$

$$0 < x \leq 1, \quad U(0) = U(1) = 0$$

The following below Figures 6, 7 and 8 shows the comparison of Approximate solution and the Exact solution. The results reflects the stability of the method for the type of singular two point boundary value problems and in particular finding the solution at very near to singular point is possible with reliability.

### CONCLUSION

The B-spline basis functions defined recursively are incorporated in the collocation method and applied the same to the singular boundary value problems. The effectiveness of the proposed method is illustrated by considering two numerical examples. The solution is compared with exact solution and found to be in good approximation.

### REFERENCES

1. Abdalkaleg Hamad and Tad Miloje Radenkovic M (2014), "A Numerical Method for Singular Boundary-Value Problems",

*Journal of Applied Mathematics and Physics*, Vol. 2, pp. 882-887.

2. Carl De Boor (1972), "On Calculating with B-splines", *Journal of Approximation Theory*, Vol. 6, No. 62.
3. C de Boor and Hollig K (1982), "B-splines from parallelepipeds", *J. Analyse Math.*, Vol. 42, pp. 99–115.
4. Curry N D Schoenberg I J (1996), "On Polya frequency functions IV: The fundamental spline functions and their limits", *J. Anal. Math.*, Vol. 17, pp. 71-107.
5. David F Rogers and Alan Adams J (xxxx), "Mathematical Elements for Computer Graphics", 2<sup>nd</sup> ed., Tata McGraw-Hill Edition, New Delhi.
6. Geng F Z and Cui M G (2007), "Solving Singular Nonlinear Second-Order Periodic Boundary Value Problems in the Reproducing Kernel Space", *Applied Mathematics and Computation*, Vol. 192, pp. 389-398.
7. Hikmet Caglar, Nazan Caglar and Khaled Elfaituri (2006), "B-spline interpolation compared with finite difference, finite

- element and finite volume methods which applied to two-point boundary value problems”, *Applied Mathematics and Computation*, Vol. 175, pp. 72–79.
8. Hughes T J R., Cottrell J A and Bazilevs Y (2005), “Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement”, *Comput. Methods Appl. Mech. Engg.*, Vol. 194(39–41), pp. 4135–4195.
  9. Joan Goh J, Ahmad Abd. Majid, Ahmad Izani Md. Ismail (2011), “Extended cubic uniform B-spline for a class of singular boundary value problems”, *Science Asia*, Vol. 37, pp. 79–82.
  10. Carl De Boor (1972), “On Calculating with B-plines”, *Journal of Approximation Theory*, Vol. 6, No. 62.10. Li Z Y, Wang Y L, Tan F G, Wan X H and Nie T F (2012), “The Solution of a Class of Singularly Perturbed Two-Point Boundary Value Problems by the Iterative Reproducing Kernel Method”, *Abstract and Applied Analysis*, pp. 1-7.
  11. Mohsen A and El-Gamel M (2008), “On the Galerkin and Collocation Methods for Two Point Boundary Value Problems Using Sinc Bases”, *Computers and Mathematics with Applications*, Vol. 56, pp. 930-941.
  12. Pandey R K and Arvind K Singh (2004), “On the convergence of a finite difference method for a class of singular boundary value problems arising in physiology”, *J. Comput. Appl. Math.*, Vol. 166, pp. 553-564.
  13. Schoenberg I J, Contributions to the problem of approximation of equidistant data.





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