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Research Paper

FIXED POINT THEOREMS IN FUZZY 2-METRIC SPACE WITH IMPLICIT MAP

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In this paper, we prove some common fixed point theorem for four and six mappings on fuzzy 2-metric spaces by using implicit relations. Our result is an extension of existing results in fuzzy 2-metric spaces.

Keywords: Fuzzy 2-metric spaces, Fixed point, Contractive mapping, Implicit relations

INTRODUCTION

The fuzzy metric space was introduced by Kramosil and Michalek in 1975. Grabiec (1988) proved the contraction principle in fuzzy metric spaces in 1988. Moreover, George and Veeramani (1994) modified the notion of fuzzy metric spaces with the help of t-norms in 1994. Gahler (1983) investigated 2-metric spaces in a series of his papers. Sharma investigated, for the first time, contraction type mappings in 2-metric spaces. Many authors have studied common fixed point theorems in fuzzy metric spaces. Some of interesting papers are Cho (1997), George and Veeramani (1994), Grabiec (1988), Kramosil and Michalek and Sharma. Cho (2006) proved a common fixed point theorem for four mappings in fuzzy metric spaces and Sharma proved a common fixed point theorem for three mappings in fuzzy 2-metric spaces. In this paper we prove a common fixed point theorem for six mappings in fuzzy 2-metric spaces by using implicit relation. Our theorem is an extension of results proved in fuzzy 2-metric spaces.

PRELIMINARIES

Definition 2.1: A triangular norm (t norm)* is a binary operation on the unit interval [0; 1] such that for all a; b; c; d $\in$ [0; 1] the following conditions are satisfied:

(1) a * 1 = a

(2) a * b = b * a

(3) a * b $\leq$ c * d; whenever a $\leq$ c and b $\leq$ d

(4) a * (b * c) = (a * b) * c

Definition 2.2: The 3- tuple (X;M; *) is called a fuzzy 2- metric space if X is an arbitrary set, * is a continuous t norm and M is a fuzzy set in $X^3 \times$

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[0; ∞) satisfying the following conditions: for all x; y; z; u ∈ X and t1; t2; t3 > 0

1. M(x; y; z; 0) = 0
2. M(x; y; z; t) = 1; t > 0 and when at least two of the three points are equal.
3. M(x; y; z; t) = M(x; z; y; t) = M(y; z; x; t) (Symmetry about three variables)
4. M(x, y, z, t_1+t_2+t_3) e" M(x, y, z, t_1)* M(x, y, z, t_2)* M(x, y, z, t_3) (This corresponds to tetrahedron inequality in 2- metric space) The function value M(x; y; z; t) may be interpreted as the probability that the area of triangle is less than t:
5. M(x; y; z; .) : [0; ∞) → [0; 1] is left continuous.

Example 2.3: Let (X; d) be a 2-metric space and denote a * b = ab for all a; b ∈ [0; 1]. For each h; m; n ∈ R+ and ∀t > 0; define M(x; y; z; t) = $\frac{ht^n}{ht^n + md(x,y,z)}$. Then (X; M; *) is a fuzzy 2-metric space.

Example 2.4: Let X be the set {1, 2, 3, 4} with 2-metric d is defined by

$$d(x,y,z) = \begin{cases} 0, & if\ x = y, y = z, z = x\ and \{x,y,z\} = \{1,2,3\} \\ \frac{1}{2} & otherwise \end{cases}$$

for each t ∈ [0;∞) , define a * b * c = abc and

$$M(x,y,z,t) = \begin{cases} 0 & if\ t = 0 \\ \dfrac{t}{t + d(x,y,z)} & if\ t > 0\ where\ x,y,z \in X \end{cases}$$

Then (X; M; *) is a fuzzy 2-metric space.

Definition 2.5: A sequence {x_n} in a fuzzy 2-metric space (X; M; *) is said to converge to x in X if and only if $\lim_{n\to\infty}$M(x_n, x, a, t) = 1; for all a ∈ X and t > 0.

Definition 2.6: Let (X;M; *) be a fuzzy 2-metric space. A sequence {x_n} is called cauchy sequence if and only if $\lim_{n\to\infty}$ M (x_{n+p}; x_n; a; t) = 1; for all a ∈ X and p > 0; t > 0.

Definition 2.7: A fuzzy 2- metric space (X; M; *) is said to be complete if and only if every cauchy sequence in X is convergent in X.

Definition 2.8: Self mapping S and T of a fuzzy 2-metric space (X; M; *) are said to weakly commuting if M(STx; TSx; z; t) ≥ M(Sx; Tx; t); for each x ∈ X and t > 0.

Definition 2.9: Self mapping S and T of a fuzzy 2-metric space (X; M; *) are said to be compatible if $\lim_{n\to\infty}$ M(STx_n; TSx_n; z; t) = 1 ∀ t > 0 whenever {x_n} is a sequence in X such that Tx_n; Sx_n → x for some x in X as n → ∞.

FIXED POINT THEOREMS IN FUZZY 2-METRIC SPACE

Definition 2.10: Suppose S and T be self mappings of a fuzzy 2-metric space (X; M; *). A point x in X is called a coincidence point of S and T if and only if Sx = Tx, then w = Sx = Tx is called a point of coincidence of S and T.

Definition 2.11: Let X be a set, f; g are self maps of X. A point x in X is called coincidence point of f and g if and only if fx = gx. We shall call w = fx = gx a point of coincidence of f and g:

Definition 2.12. Self maps S and T of a fuzzy 2-metric space (X; M; *) are said to be weakly compatible if they commute at their coincidence points that is Sx = Tx for some x ∈ X then STx = TSx:

Definition 2.13. Self maps S and T of a fuzzy 2-

metric space (X; M; *) are said to be occasionally weakly compatible (owc) if and only if there is a point x in X which is coincidence point of S and T at which they commute.

Lemma 2.14: Let X be a set, f; g owc self maps of X. If f and g have a unique point of coincidence, w = fx = gx; then w is the unique common fixed point of f and g.

IMPLICIT RELATION

Let {Φ} be the set of all real continuous function ϕ : $(R+)^6 \rightarrow R^+$ satisfying the following condition ϕ(u; u; v; v; u; u) $\geq$ 0 imply u $\geq$ v for all u; v $\geq$ [0; 1]:

Theorem 3.1: Let (X; M; *) be a fuzzy 2-metric space with * continuous t–norm. Let A;B be two self mappings of X satisfying

1. The pair (A; S) be owc.
2. For some $\phi \in \Phi$ and for all x; y; z $\in$ X and every t > 0

   ϕ{M(Ax; Ay; z; t);M(Ax; Sy; z; t);M(Ax; Sx; z; t);

   M(Ay; Sy; z; t); M(Ay; Sx; z; t); M(Sx; Sy; z; t) $\geq$ 0.

then there exist a unique point w $\in$ X such that Aw = Sw = w. Moreover w is a unique common fixed point of A and S.

Proof: Let the pair {A; S} be owc . So there are points x; y; z $\in$ X such that Ax = Sx. We claim that Ax = Ay. If not, by inequality (2),

ϕ{M(Ax; Ay; z; t);M(Ax; Ay; z; t);M(Ax; Ax; z; t);

M(Ay; Ay; z; t);M(Ay; Ax; z; t);M(Ax; Ay; z; t) $\geq$ 0

ϕ{M(Ax; Ay; z; t);M(Ax; Ay; z; t); 1; 1;M(Ay; Ax; z; t);M(Ax; Ay; z; t) $\geq$ 0.

ϕ{M(Ax; Ay; z; t); M(Ax; Ay; z; t); 1; 1; M(Ax; Ay; z; t); M(Ax; Ay; z; t) $\geq$ 0.

In view of Φ we get Ax = Ay. That is Ax = Sx = Ay = Sy.

Suppose that there is another point w $\in$ X such that Aw = Sw then by (1) we have Aw = Sw = By = Ty. So Ax = Aw and w = Ax = Sx is the unique point of coincidence of A and S: By lemma (2.14), w is a common fixed point of A and S.

To prove the uniqueness: Let w1; w2 be two common fixed points of A and S. Assume that $w_1 \neq w_2$.

ϕ(M(Aw_1; Aw_2; z; t); M(Aw_1;Aw_2; z; t); M(Aw_1;Aw_1; z; t);

M(Aw_2; Aw_2; z; t); M(Aw_2;Aw_1; z; t); M(Aw_1;Aw_2; z; t)$\geq$0

ϕM(w_1; w_2; z; t); Mw_1; w_2; z; t); M(w_1; w_1; z; t);

M(w_2; w_2; z; t); M(w_2; w_1; z; t); M(w_1; w_2; z; t)$\geq$0

ϕ(M(w_1; w_2; z; t);M (w_1; w_2; z; t); 1; 1; M (w_1; w_2; z; t); M(w_1; w_2; z; t)$\geq$0.

Therefore we get $w_1 = w_2$.

Theorem 3.2: Let (X; M; *) be a fuzzy 2-metric space with * continuous t–norm. Let A; B; S; T be four self mappings of X satisfying

1. The pairs (A; S) and (B; T) be owc.
2. For some $\phi \in \Phi$ and for all x; y; z $\in$ X and every t > 0; ϕ(M(Ax; By; z; t); M(Sx; Ty; z; t); M(Sx; Ax; z; t));

   M(Ty;By; z; t); M(Ax; Ty; z; t); M(Sx;By; z; t)$\geq$0.

then there exist a unique point w $\in$ X such that Aw = Sw = w and a unique point z $\in$ X such that Bz = Tz = z. Moreover z = w, so that there is a unique common fixed point of A;B; S and T.

Proof: Let the pairs {A; S} and {B; T} be owc. So there are points x; y; z $\in$ X such that Ax = Sx and By = Ty. We claim that Ax = By. If not, by inequality (2),

ϕ{M(Ax; By; z; t);M(Sx; Ty; z; t); M(Sx; Ax; z; t);

M(Ty; By; z; t);M(Ax; Ty; z; t);M(Sx;By; z; t)}≥0

ϕ{M(Ax; By; z; t); M(Ax;By; z; t); M(Ax; Ax; z; t);

M(By; By; z; t);M(Ax;By; z; t);M(Ax;By; z; t)}≥0

ϕ{M(Ax; By; z; t);M(Ax;By; z; t); 1; 1;M(Ax;By; z; t);M(Ax;By; z; t)} ≥ 0

In view of Φ we get Ax = By. That is Ax = Sx = By = Ty.

Suppose that there is another point w ∈ X such that Aw = Sw then by (i) we have Aw = Sw = By = Ty: So Ax = Aw and w = Ax = Sx is the unique point of coincidence of A and S. By lemma(2.14) w is a common fixed point of A and S.

And suppose that there is another point u ∈ X such that Bu = Tu then by (1) we have Ax = Sx = Bu = Tu. So By = Bu and u = By = Ty is the unique point of coincidence of B and T. By lemma(2.14) u is a common fixed point of B and T.

Assume that w ≠ u we have

ϕ{M(Aw;Bu; z; t);M(Sw; Tu; z; t);M(Sw;Aw; z; t);

M(Tu;Bu; z; t);M(Aw; Tu; z; t);M(Sw;Bu; z; t)≥0

ϕ{M(w; u; z; t);M(w; u; z; t);M(w;w; z; t);

M(u; u; z; t);M(w; u; z; t);M(w; u; z; t)} ≥ 0

ϕ{M(w; u; z; t);M(w; u; z; t); 1; 1;M(w; u; z; t);M(w; u; z; t)} ≥ 0

In view of Φ we get w = u. By lemma(2.14) z is a common fixed point of A;B; S and T.

To prove the uniqueness:

Let w_1; w_2 be two common fixed points of A;B; S and T.

Assume that $w_1 \neq w_2$.

ϕ{M(Aw1; Bw2; z; t); M(Sw1; Tw2; z; t); M(Sw1;Aw1; z; t);

M(Tw2;Bw2; z; t);M(Aw1; Tw2; z; t); M(Sw1; Bw2; z; t)} ≥ 0

ϕ{M(w_1;w_2; z; t);M(w_1;w_2; z; t); M(w1;w1; z; t);

M(w2;w2; z; t);M(w_1;w_2; z; t);M(w_1;w_2; z; t)}≥0

ϕ{M(w_1;w_2; z; t);M(w_1;w_2; z; t); 1; 1;M(w_1;w_2; z; t);M(w_1;w_2; z; t)} ≥ 0

Therefore we get $w_1 = w_2$.

Theorem 3.3: Let (X; M; *) be a fuzzy 2-metric space with * continuous tnorm. Let A;B; f; S; T; g be six self mappings of X satisfying

1. The pair (A; S); (B; T) and (f; g) be owc.
2. For some ϕ ∈ Φ and for all x; y; z ∈ X and every t > 0; ϕ{M(Ax;By; fz; t);M(Sx; Ty; gz; t); M(Ax; Sx; fz; t);

M(Ty;By; gz; t);M(Ax; Ty; fz; t);M(Sx;By; gz; t)} ≥ 0.

then there exist a unique point w ∈ X such that Aw = Sw = w and a unique point z ∈ X such that Bz = Tz = z and a unique point v ∈ X such that fv = gv = v: Moreover w = z = v; so that there is a unique common fixed point of A; B; f; S; T and g:

Proof: Let the pairs {A,S}; {B,T} and {f, g} and ff; gg be owc.

So there are points x; y; z ∈ X such that Ax = Sx and By = Ty and fz = gz:

We claim that Ax = By = fz: If not, by inequality (2),

ϕ{M(Ax;By; fz; t);M(Sx; Ty; gz; t);M(Ax; Sx; fz; t);

M(Ty;By; gz; t);M(Ax; Ty; fz; t);M(Sx;By; gz; t)} ≥ 0

ϕ{M(Ax;By; fz; t);M(Ax;By; fz; t);M(Ax; Ax; fz; t);

M(By;By; fz; t);M(Ax;By; fz; t);M(Ax;By; fz; t)}≥0.

ϕ {M(Ax;By; fz; t);M(Ax;By; fz; t); 1; 1;M(Ax;By; fz; t);M(Ax;By; fz; t)} ≥ 0

In view of Φ we get Ax = By = fz. That is Ax = Sx = By = Ty = fz = gz. Suppose that there is another point w ∈ X such that Aw = Sw then by (1) we have

Aw = Sw = By = Ty = fz = gz: So Ax = Aw and w = Ax = Sx is the unique point of coincidence of A and S: By lemma(2.14) w is the only common fixed point of A and S.

Suppose that there is another point u ∈ X such that Bu = Tu then by (i) we have Ax = Sx = Bu = Tu = fz = gz: So By = Bu and u = By = Ty is the unique point of coincidence of B and T. By lemma(2.14) u is the only common fixed point of B and T. And suppose that there is another point v ∈ X such that fv = gv then by (i) we have Ax = Sx = By = Ty = fv = gv. So fz = fv and v = fz = gz is the unique point of coincidence of f and g. By lemma(2.14) v is the only common fixed point of f and g. Assume that w ≠ u ≠ v we have

ϕ {M(Aw;Bu; fv; t);M(Sw; Tu; gv; t);M(Sw;Aw; fv; t);

M(Tu;Bu; gv; t);M(Aw; Tu; fv; t);M(Sw;Bu; gv; t) ≥ 0

ϕ {M(w; u; v; t);M(w; u; v; t);M(w;w; v; t);

M(u; u; v; t);M(w; u; v; t);M(w; u; v; t) ≥ 0

ϕ {M(w; u; v; t);M(w; u; v; t); 1; 1;M(w; u; v; t);M(w; u; v; t)} ≥ 0

In view of Φ we get w = u = v. By lemma(2.14) w is a common fixed point of A;B; f; S; T and g.

To prove the uniqueness: Let w_1;w_2 be two common fixed points of A;B; f; S; T and g. Assume that $w_1 \neq w_2$:

ϕ {M(Aw_1;Bw_2; v; t);M(Sw_1; Tw_2; v; t); M(Sw1; Aw1; v; t);

M(Tw2;Bw2;v;t);M(Aw_1;Tw_2; v; t); M(Sw_1;Bw_2; v; t) ≥ 0

ϕ {M(w_1;w_2; v; t);M(w_1;w_2; v; t);M(w1;w1; v; t);

M(w_2;w_2; v; t);M(w_1;w_2; v; t);M(w_1;w_2; v; t)} ≥ 0

ϕ {M(w_1;w_2; v; t);M(w_1;w_2; v; t); 1; 1;M(w_1;w_2; v; t);M(w_1;w_2; v; t)} ≥ 0.

Therefore we get $w_1 = w_2$.

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